

THE MATHEMATICAL, PROBABILISTIC AND COMPUTATIONAL GENERATORS OF DISCRETE PROBABILISTIC DISTRIBUTIONS APPLIED TO MEDICAL PHYSICS

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Abstract— Despite the binomial distribution (BD) and its limiting case, the Poisson distribution (PD) are probabilistic functions (PFs), they are mathematical derivations, where little probabilistic foundations were used in their formulations. These PFs have been overestimated and irrationally used in many scientific/technical fields.

This work discussed three practical examples. One of them shows the expected values for “ k ” trails of a stochastic process characterized with a success probability “ p ”. Another examples are related with use of the BD as an excellent-mathematical generator of NTCP(xi) discrete probabilistic distributions; and with use of BD/Poisson distribution (PD) in the nuclear medicine for evaluating probabilities of measurements of a long-lived radioactive sample as a random variable N , as well as use of cumulative distribution functions for evaluating probabilities of some interval of N , and for determining with iterative calculations confidence intervals. The BD($x;2,p$) terms are probabilistically obtained; as well as a MatLab application was developed, which generates random discrete probabilistic distributions based on probabilistic foundations. The irrational use of the PD in the derivation of the Poisson-based TCP model has been described.

This work will help medical physics community to understand: 1) How the BD and PD were derived; 2) What really the BD and PD are; 3) How one should use the BD; 4) The PD is not a new PF, but the own BD with simplifications valid for some values of BD parameters, and changes of variable and parameters; and 5) Given the essential condition for a PF is not satisfied in the PD for some values of its parameter, also we can say that: The PD is not a PF. For these reasons, the PD($x;\mu$) could be replaced with the BD($x;Xmax,p$), where $Xmax$ is the possible outcomes of a stochastic process, $Xmax=n$ and $p=\mu/Xmax$.

Keywords—Binomial distribution, Poisson distribution, Computational simulation, TCP, Probability

I. INTRODUCTION

The SMp(x) of [1] is a probabilistic function (PF) that lets us generating probability density functions and discrete probabilistic distributions (DPDs). Although SMp(x) has six parameters, up to five of them are independent given its condition of PF.

The BD and PD are analytical functions that generate DPDs. These functions have mathematical origins, and have been over-estimated in their applications. Besides, they have been irrationally used in the ionizing radiation field for deriving the Poisson-based tumor control probability (TCP) model and describing the interactions of ionizing radiation with living tissues.

Despite the associated elaboration of a computational application is one of our objectives; in this study there are others more important purposes, such as

1- To show the probabilistic way of obtaining the BD($x;2,p$) terms.

2- To show the irrational use of the PD in the derivation of the Poisson-based TCP model.

3- To show the PD is not a new probabilistic function, nor its parameter is a new one. PD is the own BD($k;n,p$) with some mathematical simplifications valid only for some values of parameters n and p , and change of the BD variable k by x , as well its parameters n and p by λ or μ as the product of them; i.e. $\lambda=\mu=n*p$. For these reasons, the PD($x;\mu$) should be replaced with the BD in the form BD($x;Xmax,p$), where $Xmax$ is the possible outcomes of a stochastic process, $Xmax=n$ and $p=\mu/Xmax$.

4- To establish the $BD(x;X_{max},p)$, $SMP(x;X_{max},p)$ and $SMP(x;X_{max})$ simulator as ways of obtaining DPDs.

We hope the student/teacher/researcher communities will understand that BD is only an excellent- mathematical generator of DPDs, and PD is not a new probabilistic function, but the own BD, as well as λ and μ are not new parameters, but the product of BD ones; i.e. $\lambda=\mu=n*p$. This understanding could avoid the overestimated, confused and irrational use of BD and PD in different fields.

It is very essential to understand that if a stochastic process has a success probability p , $p=K/N$ (K : Success events, and N : Total of events) and for n trials, the ratio k/n (k : Expected success events) is equal to or approximately equal to p . This understanding will let you realizing of the little probabilistic and no practical importance of the $BD(k;n,p)$.

1.1 The binomial distribution

The binomial expression of the Eq. (2) mathematically provides a sum of $\binom{n}{k}$ of $p^k(1-p)^{n-k}$ as result of replacing $a=p$ and $b=1-p$ in the Eq. (1), the binomial theorem, with which one can determine the series of sums for the power n of a sum of two number a and b . Here p is a parameter < 1 and $k=0,1,2,\dots,n$

$$(a + b)^n = \binom{n}{0}a^0b^{n-0} + \binom{n}{1}a^1b^{n-1} + \dots + \binom{n}{n}a^nb^{n-n} \quad (1)$$

$$(p + (1-p))^n = 1 = \binom{n}{0}p^0(1-p)^{n-0} + \binom{n}{1}p^1(1-p)^{n-1} + \dots + \binom{n}{n}p^n(1-p)^{n-n} \quad (2)$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (3)$$

The BD generates DPDs in the interval $[0;n]$ varying its parameter p , where n is the other mathematical parameter.

The computational simulations of the BD generates a DPD of stochastic processes with three or more possible outcomes from many (>10000) simulations of a stochastic process with success probability p and n trials. This was computationally demonstrated with the simulator of [2].

1.2 The Poisson distribution.

Siméon Denis Poisson, the creator of the PD had two important merits: 1) Mathematically simplifying the BD expression; and 2) Replacing the BD variable k by x in his simplified expression; where they left to mention k as success trails and p as success probability, but variable x and parameter μ as the value of x with the maximum probability. μ is not a new parameter, but the product of the two BD parameters; i.e. $\lambda=\mu=n*p$.

The PD was determined as follows

$$\lim_{n \rightarrow \infty} BD(k; n, p) = PD(k; \lambda) = \frac{e^{-\lambda} \lambda^k}{k!} \quad (4)$$

and employed for calculating probabilities of a random discrete variable X as

$$PD(x; \mu) = \frac{e^{-\mu} \mu^x}{x!} \quad (5)$$

1.3 The irrational use of the BD and PD in the ionizing radiation field

In the ionizing radiation field, the BD and PD are irrationally used for deriving some probabilistic models and concepts.

The use of Poisson statistics (PS) in TCP models has led to a negative-exponential expression in [3] and [4]. Also, the cell survival (S) has been described with the PS in these same references. The ways of describing S with the PS is probabilistically very complicated. Really, S is a probabilistic complement of the cell kill (K); i.e. $K=1-S$, and K can be modelled with the $SMP(x)$ function of [1] as a stochastic effect type $SMP P1$.

For the formulation of TCP model of [3], the Eq. (5) was used and transformed as

$$PD = e^{-\mu} \quad (6)$$

that is result of considering the Poisson independent variable x as number of tumor clonogens, and equal to zero.

As expressed in [3]: “The Poisson probability of there being no surviving cells in a population of like tumors after a fractionated treatment is given by”

$$TCP = e^{-Ns} \quad (7)$$

$$N_s = N_0 S \tag{8} \quad P_{k=1} = EV1/rtrials \tag{16}$$

where N_0 : number initial of tumor clonogens.

S : The cell survival probability, which is modeled with the well-known linear-quadratic cell survival model for a fractionated radiation treatment.

where $EV1$ is equal or approximately equal to the number of trials with one success in two trials; $rtrials$: number of times is repeated the two trials.

II. RESULTS

II.1 Probabilistic determination of the $BD(x;2,p)$ terms.

The $BD(2;2,p)$ term associated to the probability $P_{k=2}$ is defined as

$$P_{k=2} = EV2/rtrials \tag{9}$$

where $EV2$ is equal or approximately equal to the number of trials with two successes in two trials; $rtrials$: number of times is repeated the two trials.

$$K_1 = p * rtrials \tag{10}$$

$$K_2 = p * rtrials \tag{11}$$

where K_1 : Equal or approximately equal to amount of successes in the first trials; and K_2 : The same to K_1 in the second trials.

For many $rtrials$ (>10000) $K_1 \cong K_2$, and $p * K_2$ is equal or approximately equal to the number of times that two trials will produce two successes; i.e. $p * K_2 = EV2$, substituting $EV2$ in the Eq. (9), and K_2 from the Eq. (11), we obtain that

$$P_{k=2} = p * p * rtrials / rtrials \tag{12}$$

$$P_{k=2} = p * p \tag{13}$$

$$P_{k=2} = p^2 \tag{14}$$

The $BD(0;2,p)$ associated to probability $P_{k=0}$ is determined with a similar procedure employed in the term $BD(2;2,p)$, but in this analysis the failures should be considered, instead of successes. For these reasons,

$$P_{k=0} = (1 - p)^2 \tag{15}$$

The $BD(1;2,p)$ associated to probability $P_{k=1}$ is defined as

$$K_1 = p * rtrials \tag{17}$$

$$K_2 = p * rtrials \tag{18}$$

where K_1 is equal or approximately equal to amount of successes in the first trials; and K_2 : The same to K_1 in the second trials.

For many $rtrials$ (>10000) $K_1 \cong K_2$, and $(1-p) * K_2 = K''$ is equal or approximately equal to the number of successes in the second trial that have a failure in first trial, $(1-p) * K_1 = K'$ is equal or approximately equal to the number of successes in first trials that have a failure in second trials, and $EV1 = K' + K''$, and substituting K_1 in K' , and K_2 in K'' of the Eq. (17) and Eq. (18) respectively, then

$$K' = (1 - p) * p * rtrials \tag{19}$$

$$K'' = (1 - p) * p * rtrials \tag{20}$$

Substituting $EV1$ in the Eq. (16),

$$P_{k=1} = \frac{K' + K''}{rtrials} \tag{21}$$

Using the Eq. (19) and Eq. (20),

$$P_{k=1} = \frac{(1-p) * p * rtrials + (1-p) * p * rtrials}{rtrials} \tag{22}$$

$$P_{k=1} = 2 * (1 - p) * p \tag{23}$$

II.2 Random generation of discrete probabilistic distributions

II.2.1 Description of the codes

This work has developed a MatLab computational tool, which generates random discrete probabilistic $SMP(x;Xmax)$ distributions, where $Xmax$ is number of possible cases. This application is available in the “GenDPD” project of <https://gitlab.com/tfrometa>. The generation of these distributions probabilistically satisfies that $\sum SMP(x_i; Xmax) = 100\%$ always.

II.2.2 Reproducibility

At the application, the input value (IV) X_{max} appears in yellow color, while outcome of the sum of simulated probabilities appears in green. One should press the “Enter” key placed at X_{max} field for introducing its value into the application. The Figure 1 shows a generated discrete probabilistic Smp ($x;5$) distribution.

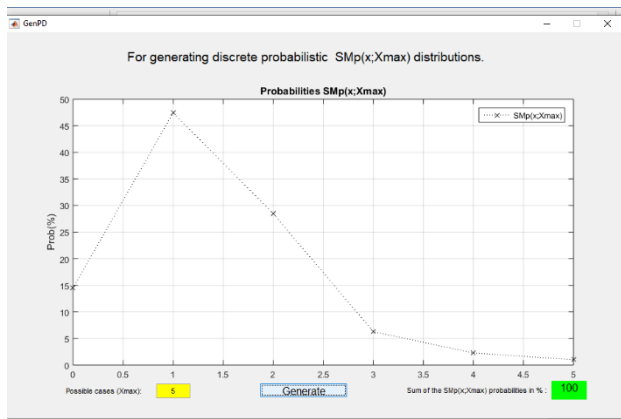


Figure 1. It is shown that a random-generated discrete probabilistic distribution and its sum as $\sum_{i=0}^5 Smp(x_i; 5) = 100\%$.

The steps for the execution of this module are: a) Introduce parameters X_{max} ; and b) Press the “Generate” button

III. DISCUSSION

III.1 The binomial and Poisson distributions

Really, PD is a mathematical simplification of the BD. While PD has simplified the BD expression, the methodologies employed in the “Binomial” and “Poisson” modules of [2], are very similar. For these reasons, in the new computational simulator of the “SimPD1” project of <https://gitlab.com/tfrometa>, the “Poisson” module has been eliminated, and the “Binomial” module has been renamed as “Binomial-Poisson”, which generates probabilistic Smp($x;X_{max},p$) distributions, where $p=\mu/X_{max}$ and $X_{max}=n$. The Figure 2 and Figure 3 are results obtained by the computational simulator developed in [2] and one of the two applications developed in this study; and show the whole

coincidence of the simulated Smp($x;X_{max},p$) distributions with the BD.

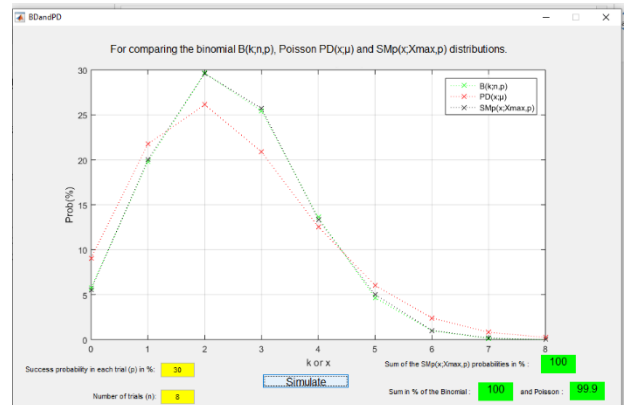


Figure 2. It is shown that Poisson generates a proper for describing this discrete distribution, since $\sum_{i=0}^8 PD(x_i) \approx 100\%$.

While the BD results are obtained with mathematical procedures, the simulated Smp distributions are obtained by means many simulations (>10000) of a stochastic process with a success probability p and repeated n trials.

The BD and Smp distributions are always DPDs in the interval $[0;X_{max}]$, however for some values of parameters n and p , as the Figure 2 shows, the PD is good approximation of the BD and generates DPDs, but for others, as the Figure 3 illustrates, PD is not good approximation of the BD and does not generate DPDs because of this PD does not satisfy the essential condition for the probabilistic functions, where $sum(PD(x_i))$ must be equal to 100%.

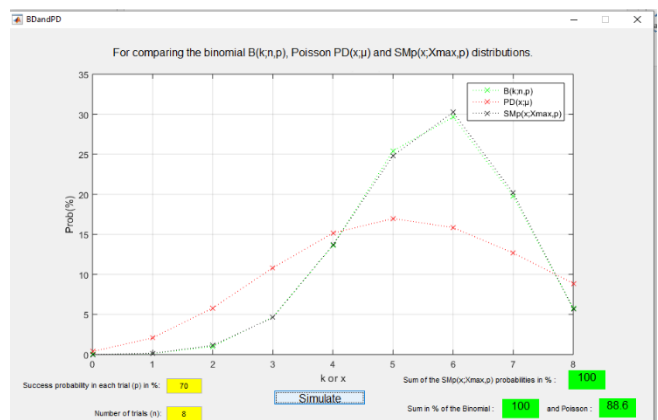


Figure 3. It is shown that this PD does not generate a DPD in this interval, and is not good approximation of the BD. There a difference of 11.4% in the sum of the $P(x_i)$; i.e. $\sum_{i=0}^8 PD(x_i) \neq 100\%$.

Although the BD expression coincidentally is a DPD of k successes of many repeated (>10000) n trials of a stochastic

process (SP) with success probability p , really when a SP is characterized with a success probability p , it implies that:

a) p was obtained as $p=K/N$, where K is the number of success results and N is the total of events.

b) Each trial has a success probability p .

c) In n trials, the ratio of k successes and n trials should be equal or approximately equal to p ; i.e. $\frac{k}{n} \cong p = K/N$. For example: When a homogeneous population of 100 patients are treated with a same radiation oncology treatment, and 60 of them were cured; we say this treatment has a tumor control probability (TCP) equal to 60%; and if n new patients are treated later, the relationship between n and expected patients cured (k) will be

| | | | | | | | |
|-----------------------------|---|----|----|----|----|----|-----|
| New patients (n) | 5 | 10 | 25 | 40 | 70 | 80 | 100 |
| Expected patients cured (k) | 3 | 6 | 13 | 24 | 42 | 48 | 60 |

d) There is no need of creating a second and new probability $P(k;n,p)$ dependent of p , like the $BD(k;n,p)$.

III.2 The Poisson distribution in ionizing radiation field

The Poisson-based TCP model is wide employed in the field of the radiation treatments, how is shown in [5] and [6].

The criteria of “no surviving cells” or “100% killed cells” are proper for determining TCP. In fact, this condition was used in the first radiobiological simulator of [7].

Although the “no surviving cells” is a good criterion associated to TCP, it is not correct to associate the Poisson independent variable x with the number of tumor clonogens, and the Eq. (5) with TCP.

The BD can be used for describing or assuming $NTCP_i$ $i=0:n$ (n : number of complications) that involves $NTCP_s$ (Normal tissue complication probabilities) and $NTCP_0$ (Normal tissue non-complication probability). $NTCP_0$ is a new probabilistic metric associated to evaluations of safety in whatever risky activity, like radiation oncology therapy. $NTCP_0=NTCP_0$ and is probabilistic complement of the total $NTCP$ ($TNTCP$); $NTCP_0=100\%-TNTCP$, and $TNTCP =$

$\sum_{i=1}^n NTCP_i$. $TNTCP$ is a new probabilistic metric too. $NTCP_0$ is associated to the safety, and $TNTCP$ is associated to toxicity, how is described in [8].

III.3 Teaching/learning importance of this work.

This work provides the following teaching lessons:

a)- The BD and PD have been overestimated and irrationally used in different fields, like the ionizing radiation one.

b)- The BD is an excellent-mathematical generator of DPDs, and coincidentally is associated to the probabilities $P(k;n,p)$ only for many repeated n trials (> 10000).

c)- The $PD(x;\mu)$ is not a new probabilistic function, but the own BD with a change of variables and simplifications valid for determined values of the BD parameters (n and p); and its parameters (λ and μ) are only the product of n and p . A simple change of variable k by x does not generate a new function. For these reasons, the $PD(x; \mu)$ should be replaced with the $BD(x;Xmax,p)$. Even, given the essential condition for a PF is not satisfied in the PD for some values of its parameter (See Figure 3), also we can say that: The PD is not a PF.

d)- The probabilistic generation of the $BD(x;2,p)$ terms.

e)- The irrational use of the PD in the derivation of the Poisson-based TCP model.

f)- Additional to mathematical way of generating DPDs, there are other methods, such as the probabilistic and computational.

g)- Unnecessary use of the $P(k;n,p)$ probabilities, like the BD, for stochastic processes are characterized with a success probability p . In the Section III.1, with a practical example, we show what means whether a stochastic process is characterized with a success probability p . For this reason, the BD should be used as $BD(x;Xmax,p)$, where p is simply a mathematical parameter. For choosing the binomial parameter p , one should take into account that: 1) if $p \ll 0.5$, the $BD(0;Xmax,p)$ is the event with maximum probability (EwMP); 2) if $p < 0.5$, one of the $BD(x \neq 0;Xmax;p)$ is the EwMP, and $BD(0;Xmax,p) \gg 0\%$; if $p \approx 0.5$, one of the $BD(x \neq 0;Xmax,p)$ is the EwMP, and $BD(0;Xmax,p) > 0\%$; and 3) if $p > 0.5$, one of the $BD(x \neq 0;Xmax,p)$ is the EwMP, and $BD(0;Xmax,p) \approx 0\%$.

h)- While mathematically BD does not show, our applications show that BD is associated to k success events of n trials of a many times (>10000) simulated stochastic process with a success probability p .

All new knowledges have been validated with theoretical demonstrations or with computational tools.

These new knowledges could avoid the overestimated, confused and irrational use of the BD and PD by the student/teacher/researcher communities in different fields. Take into account that the BD and PD are elemental topics in many teaching materials of the statistics, and are probabilistic functions that have been well-established for more than 200 years.

Our work shows that there are three ways of obtaining DPDs: mathematical, probabilistic and computational, used respectively in the $BD(x;X_{max},p)$, $SMp(x)$ function, and SMp simulators. Although these ways are different, they generate similar DPDs, which can be used for describing or assuming discrete stochastic variables with three or more possible outcomes.

When a process is deterministic, this will occur always or will not occur never. While if a process is stochastic, the expected ratio k/n of the number of success events (k) and total of them (n) will be equal or approximately equal to the success probability p , which is generally determined as $p=K/N$; where K : success observations, and N : Total of them. For these reasons, the $BD(k;n,p)$ has little probabilistic and does not have practical importance. The BD is a mathematical exercise that is result of evaluating respectively the variables a and b of the binomial theorem by p and $1-p$; and its expression is a sum of the “ $n+1$ ” elements of a DPD.

The Figure 4 illustrates a hypothetical example of a $NTCP(x_i)$ DPD= $BD(x;4,0.3)$ for describing or assuming the probabilities of late complications discussed in [9]-[10], and associated to a chest radiation treatment involving complications of two OARs: heart and lung. The $NTCP_0=NTCP(0)=24\%$. This value increases if prescribed dose ($D=nd$; n : Number of fractions, and d : Dose per fraction) decreases, and vice versa, as result of variations of d for a treatment with a constant n ; or variations of n for a constant d . The $NTCP_0$ value increases if D decreases, and vice versa, how is shown by the four arrows on the right-side of the y-axis of the Figure 4.

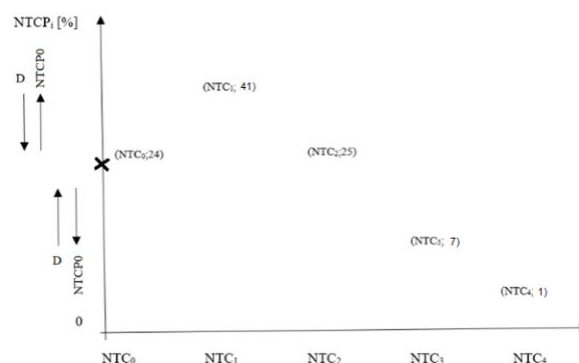


Figure 4. Hypothetical example of a $NTCP_1$ or $NTCP(x)$ DPD equal to $BD(x;4,0.3)$ for describing or assuming the probabilities of late complications associated to a chest radiation treatment. Abbreviations: **D**: Prescribed dose; **NTC₀** No complication; **NTC₁** Congestion heart failure; **NTC₂** Ischemia; **NTC₃** Coronary artery disease; **NTC₄** Pneumonitis. The $NTCP_0=NTCP_0=24\%$ is represented by a **x**.

Whatever radiation therapy or radiation activity with similar circumstances has its own $NTCP(x_i)$ DPD caused by the affected OARs and others have physiological relationships with these.

Probabilistically one can say for a stochastic process (SP) with only one outcome; for example, if a radiation treatment has a $TCP=60\%$, that there is not tumor control $100\%-60\%=40\%$; while one can say for a SP with more than one outcome, like late normal tissue complications (NTC), that this SP have a $NTCP(x_i)$ DPD, where $NTCP_0$, i.e. the probability for non-complications is $NTCP(0)$.

It is very important to know how the BD and PD were derived for understating how the BD should be used. These functions are not only mathematical, but they were created as probabilistic functions, which must satisfy the essential condition of a DPD: $\sum BD(x_i)=100\%$ and $\sum PD(x_i)=100\%$. The PD is a discrete function, how is shown by its denominator called factorial that is defined only for nonnegative integer values.

Another example taken from [11] that shows the use of BD and PD is the following: “Suppose that a long-lived radioactive sample is counted repeatedly under supposedly identical conditions with a properly operating counting system. Because the disintegration rate of the radioactive sample undergoes random variations from one moment to the next, the numbers of counts recorded in successive measurements (N_1, N_2, N_3 , etc.) are not the same. Given that different results are obtained from one measurement to the next, one might question if a “true value” for the measurement actually exists. One possible solution is to

make a large number of measurements and use the average as an estimate for the “true value.””.

The Figure 5 is the graphical representation of [11] for showing the use of the PD; but its DPD can be described with $BD(N;20,0.5)$ or $PD(N;10)$. These DPDs let us evaluating probabilities of each value of N ; and their cumulative distribution functions allow to determine probabilities of intervals, like $m \pm \Delta N$ as

$$P(N > m - \Delta N \ \& \ N < m + \Delta N) = \sum_{m-\Delta N}^{m+\Delta N} BD(N; 20,0.5) \quad (24)$$

or

$$P(N > m - \Delta N \ \& \ N < m + \Delta N) = \sum_{m-\Delta N}^{m+\Delta N} PD(N; 10) \quad (25)$$

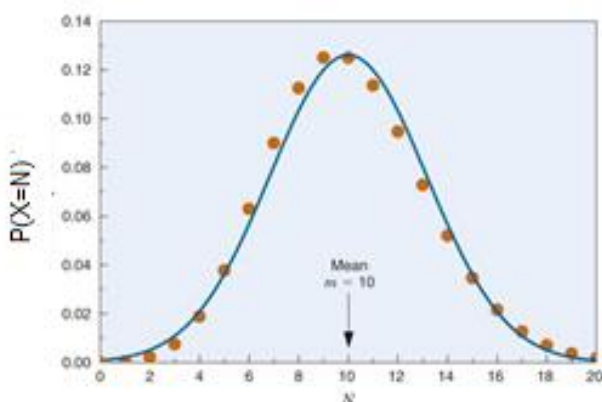


Figure 5. Graphical representation of [11] for showing the use of the Poisson distribution with expression $PD(N;10)$.

The probability of N equal to the mean ($m=10$) is equal to $BD(10;20,0.5)=12.51\%$; and $P(N>6 \ \& \ N<14)$; i.e. $\Delta N=4$, is equal to 85%. Another important value of a DPD is the most likelihood (ML). In our case, as a symmetric function $m=ML$.

Using interactive calculations one can determine a confidence interval (CI). For example, $CI=95\%$, $\Delta N \approx 5$.

IV. CONCLUSION

This study has let probabilistically obtaining the $BD(x;2,p)$ terms; developing a computational generator of random DPDs based on probabilistic foundations, and showing the irrational use of PD in the derivation of the Poisson-based TCP model.

The BD and generated SMp distributions could be used for describing or assuming DPDs, like $NTCP_i$ ($i=0:nc$, nc :

number of complications) in the radiation oncology therapies, which includes $NTCP_0$ and $NTCPs$. $NTCP_0$: Normal tissue non-complication probability; and $NTCPs$: Normal tissue complication probabilities.

In the way that BD was formulated, one cannot say that its DPD is associated to probabilities of k successes n trails of a stochastic process with a success probability p . Only with methods, like computational simulations this association is demonstrated. Also, based on the BD formulation, one should not treat to p as a probabilistic parameter, but as mathematical one, where $p < 1$. In the computational tools, p is considered as a success probability.

We have probabilistically derived the $BD(x;2,p)$ terms, and computationally demonstrated that BD is DPD of k success events of n trails (>10000) of a stochastic process with success probability p .

Given that: a) The current high computational and technological degree is a situation that lets going without of a simplification of the BD expression; b) Contrary to the BD, PD is not always an acceptable DPD for an interval $[0;Xmax]$ nor a good approximation of the BD; c) Very simple relationships among the parameters n , p and μ ; and d) the $BD(x;Xmax,p)$ can play a better role than $PD(x;\mu)$, where is easily determined p from μ as $p=\mu/Xmax$; we propose the following:

1- Using as generators of DPDs to:

- As a mathematical method, the BD as $BD(x;Xmax,p)$; i.e. the BD with change of variable k by x , and parameter n by $Xmax$.
- As a probabilistic method, the $SMp(x)$ function
- As a computational methods, the SMp simulator that generates the $SMp(x;Xmax,p)$ distributions for many repeated n trials (> 10000), and the SMp simulator that generates random $SMp(x;Xmax)$ distributions

2- Analyzing the possibility of replacing the $PD(x;\mu)$ with the $BD(x;Xmax,p)$, where $Xmax=n$ and $p=\mu/Xmax$. Due to all previously said, the use of PD will become unnecessary.

The dissemination of the new and elemental knowledges provided by our study will lead change of statistics courses involving the BD and PD topics, and will let the students a better understanding of BD and PD, as well as they will be provided of other tools for generating DPDs, such as

probabilistic and computational, different to the mathematical employed in the BD.

V. REFERENCES

1. Frometa-Castillo T. (2018) The $SMP(x$ or $y;PXmin,Xmax,ML,p1,p2,Max)$ a probabilistic distribution, or a probability density function of a random variable X , Chapter 48 of 16th International Conference on Information Technology-New Generations (ITNG 2019), Springer Nature Switzerland AG.
2. Frometa-Castillo T, Pyakuryal A, Wals-Zurita A, Mesbahi A (2020) Computational Simulations of Similar Probabilistic Distributions to the Binomial and Poisson Distributions. *Preprints*, 2020010065. doi: 10.20944/preprints202001.0065.v1).
3. Chapman JD, Nahum AE (2015) Radiotherapy treatment planning Linear-Quadratic Radiobiology. Taylor & Francis Group, LLC.
4. Dawson A, Hillen T (2006) Derivation of the tumour control probability (TCP) from a cell cycle model. *Computational and Mathematical Methods in Medicine*, 7:2-3, 121-141. DOI: 10.1080/10273660600968937.
5. Fiorino C, Broggi S, Fossati N. et al. (2015) Predicting Clinical Outcome Following Post Prostatectomy Radiation Therapy: A Poisson-Based Tumor Control Probability (TCP) Model Based on a Large Multi-institutional Series, *International Journal of Radiation Oncology Biology Physics*, Volume 93, Issue 3.
6. Schinkel, C, Stavreva N, Stavrev P, et al. (2007) Functional form comparison between the population and the individual Poisson based TCP models, *Radiol Oncology*,41(2); 90-98.
7. Frometa-Castillo T, Pyakuryal A, Piseaux-Aillon R (2019) Simulator of radiation biological effects in tumor in order to determinate the tumor control probability; *Informatics in Medicine Unlocked*; 16. <https://doi.org/10.1016/j.imu.2019.100217>.
8. Frometa-Castillo T, Pyakuryal A, Wals-Zurita A, Mesbahi A. (2020) Proposals of models for new formulations of the current complication-free cure (P+) and uncomplicated tumor control probability (UTCP) concepts, and total normal tissue complication probability of late complications, *Int J Radiat Biol*; 96(7):847-850. DOI: 10.1080/09553002.2020.1741722.
9. Marks LB, Bentzen SM, Deasy JO, et al. (2010) Radiation dose-volume effects in the lung, *International Journal of Radiation Oncology • Biology • Physics*, Volume 76, Issue 3, S70 – S76. DOI: <https://doi.org/10.1016/j.ijrobp.2009.06.091>.
10. Gagliardi G., Constine L.S., Moiseeko V., et al. (2010) Radiation dose-volume effects in the heart, *Int J. Radiation Oncology Biol. Phys.* Vol. 76 No. 3, Supplement, pp. S77-S85.
11. Nuclear Counting Statistics, Radiology Key at <https://radiologykey.com/nuclear-counting-statistics/>

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